Parametrization of the equation of state and the expanding universe

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The structure of the equation of state ω could be very complicate in nature while a few linear models have been successful in cosmological predictions. Linear models are treated as leading approximation of a complete Taylor series in this paper. If the power series converges quickly, one can freely truncate the series order by order. Detailed convergent analysis on the choices of the expansion parameters is presented in this paper. The related power series for the energy density function, the Hubble parameter and related physical quantities of interest are also computed in this paper.

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I. INTRODUCTION

Recently there have been advances in our abilities in cosmological observations for the quest of exploring the expansion history of the universe. It carries cosmology well beyond determining the present dimensionless density of matter Ω_m and deceleration parameter q_0 [1]. It seems possible to reconstruct the entire function $a(t)$ representing the expansion history of the entire universe. Earlier on, cosmologists sought only a local measurement of the first two derivatives of the scale factor a, evaluated at a single time t_0. Attempt will be made trying instead to map out the function determining the global dynamics of the universe in the near future. One notes that many qualitative elements of cosmology follow directly from the structure of the metric[2]. Therefore, deeper understanding of our universe requires knowledge of the dynamics of the scale factor $a(t)$ echoing the transition of energy between components from the epoch of radiation domination to that of matter domination. This is also known to be a key element in the growth of density perturbations into large structure. Indeed, many proposed observation will be able to probe the function a(t) more completely throughout all ages of the universe [3, 4, 5, 6].

A number of promising methods are being developed including the magnitude-redshift relation of Type Ia supernovae. The goal of mapping out the recent expansion history of the universe is well motivated. The thermal history of the universe, extending back through structure formation, matter-radiation decoupling, radiation thermalization, primordial nucleosynthesis, etc. is very important in the study of cosmology and particle physics, high energy physics, neutrino physics, gravitational physics, nuclear physics, and so on [7].

Expansion history of the universe is similarly a very promising research focus with the discovery of the current acceleration of the expansion of the universe. This includes the study of the role of high energy field theories in the form of possible quintessence, scalar-tensor gravitation, higher dimension theories, brane worlds, etc in the very recent universe. The accelerated expansion is also important to the possible fate of the universe [8, 9, 10]. It is hence important to obtain the magnitude-redshift law relating to the scale factor-time behavior $a(t)$ from these supernova observations with the proposed Supernova/Acceleration Probe mission [11].

Therefore, the study of modelling different equation of state (EOS) derived from different theories plays an important role in the study of the recent expansion history of our universe. The study of dark matter and possible contribution from different combinations of different theories including the SUGRA and string theory will become important also in the near future once the LHC experiment starts. Therefore all practical methods to study its impact on the evolution history of our universe deserve full attentions. In fact, there have been proposals to parameterize different candidate models that are helpful in extracting important physics with the help of several different linear parameterizations of parameter $z/(1+z)$ or z for the corresponding equation of state function ω [12, 13, 14, 15, 16, 17, 18, 19, 20]. These linear models are shown to be very successful in recreating various physical functions.

Linear models, e.g. $\omega = \omega_0 + \omega_1 y$ with $y = z/(z+1)$, should be thought of as leading approximation of a complete Taylor series. If this is so, result derived from higher order in y, e.g. $O(y^2)$ terms, should only provide marginal contribution and hence can be ignored. If the higher order contribution is appreciable, one should be very careful in dealing with the truncated series. As a practical analysis, one should compute all related physical functions also as a power series order by order to obtain reliable observable according to the precision requirement. Otherwise, unphysical contributions may build up and leave the final result invalid. Fortunately, this complication may be unnecessary if the higher order contributions are small as compared to the leading order contribution.

Therefore, we try to extend the parametrization further to include the effect of higher orders of the series expansion results. Related physical quantities are treated carefully order by order in order to extract more reliable information from these expansions. One also tries to determine the expansion coefficients from the fitting of the measured Hubble parameters or energy density. As a result, one may reconstruct the series expansion of the equation of state and probe the nature and origin of the matter sources.

Finally, we will also present an error analysis to find the range of convergence and possible error control for a meaningful truncation. For example, the linear model given by $\omega = -0.82 + 0.58y$ fits the SUGRA result to a good precision at large z even the linear model is already off 27% when $z \sim 1.7$[19]. One can show that the next leading expansion coefficient with $\omega_2 y^2 < 0.1y^2$ only provides at most 20% error even when the redshift is close to 1100. As a result, one can show that a reliable truncation is possible when the higher order expansion coefficient is small for all range $y = z/(z+1) = 0 \to 0.9991$ up to the last scattering surface when $z \sim 1100$. Therefore, it does not matter much whether one sums up all power of y coupled to the leading coefficients ω_0 and ω_1. In addition, the linear model $\omega = -0.4 + 0.11y$ approximates the inverse power law model very well at small y[19]. Similar analysis will also be shown in this paper for comparison.

In addition, the evolution of the equation of state ω of the physical universe could be very complicate. Indeed, the equation of state can be described by $\omega = p/\rho = (\dot{\phi}^2/2 - V)/(\dot{\phi}^2/2 + V)$ for many different effective theory with effective potential V and scalar field ϕ. For example, $V_{INV} \sim \phi^{-\alpha}$ and $V_{SUGRA} \sim \phi^{-\alpha} \exp[\phi^2]$ for inverse power law models and SUGRA models respectively[19]. The nature of our physical universe may also consist of many other combinations of scalar fields contributions. It is therefore important for us to figure out which model plays the most important role in the expansion history of our universe. The linear model or leading approximation appears to be of help in determining the essential part of the nature of the equation of state and its corresponding effective theory. Therefore a more detailed analysis on the convergence properties of various approximated models is very important as a research topic.

Since one is not sure about the nature of the function ω, the Taylor series expanding $\omega(y) = \sum_n \omega_n y^n$ into series sum with expansion parameter ω_n may also be used to determine leading expansion parameter with the help of future measurement. By fitting the Hubble parameter measurements one could determine leading terms of the expansion coefficients ω_n. As a result, one would be able to reconstruct the recent evolution of the function ω to the better precision. We will present the complete analysis in this paper. Since future observations will be able to provide tight constraint for only a few parameters in the near future, the material provides in this paper can be used to determine these parameters in a more reliable details. Hopefully, future development of observation tools may provide deeper insight to determine more constraints in these theories.

We will present, in section II, a Taylor series expansion of some physical models. One will explain the choices of the expansion variables which come from the integration involving $dz/(1+z)$. The convergence range of these expansion variables will also be presented. In section III, we will also compute leading terms of the energy density function ρ, the Hubble parameter H, and the conformal time η for these expansions in order to discuss the difference between these approaches.(cf. [18] and Fig. 1 of [19]). Due to the fact that the parameter $z/(1+z)$ has a larger naive convergent range, this parameter could be more useful in large z expansion. We also compute and analyze the convergence range of a linear model in section IV.

II. TAYLOR SERIES OF THE EQUATION OF STATE

The equation of state is defined by the relation $\omega = p/\rho$. In addition, the field equations of the universe can be shown to be:

$$d \ln \rho(z) = 3(\omega + 1) d \ln(1+z), \tag{1}$$

$$H^2 = \frac{8\pi G}{3}\rho, \tag{2}$$

for a flat FRW universe with $1+z$ the redshift defined via $a/a_0 = 1/(1+z)$. Various linear models [12], [13] have been suggested for the EOS in the literatures. We will try to point out in this paper the relations between these linear models and go further to obtain a more complete leading-order expansions.

Taylor expansion is known to be one of the best ways to extract leading contributions from a generic theory with the help of a power series expansion of some suitable field variables. The Taylor series is normally convergent quickly depending on the structure of the expansion coefficients. The power series will, however, converge quickly if the range of variable is properly chosen. Hence it is important to choose an appropriate expansion variable for the purpose of our study.

For example, one may expand the EOS, assuming to be a smooth function for all z, as a power series of the variable z around the point $z = 0$. This will lead to the power series expansion:

$$\omega(z) = \sum_{n=0}^{\infty} \frac{\omega^{(n)}(z=0)}{n!} z^n. \tag{3}$$

Here the summation of n runs from 0 to ∞ and $f^{(n)}(z) = d^n f(z)/dz^n$ for any function $f(z)$. From now on, the range of summation will be omitted for convenience throughout this paper unless it is different from the range from 0 to ∞.

One sees clearly that there are three useful and practical ways to expand the equation of state: z, $1/(1+z)$, and $\ln(1+z)$, from the structure of the conservation law given by Eq. (1). It turns out that the expansion parameters will become z, $-z/(1+z)$, and $\ln(1+z)$ respectively if one tends to expand the function ω about the point $z=0$. In practical, we will expand the second choice with the parameter $z/(1+z)$ instead of $-z/(1+z)$ for convenience.

In this section, one will first expand the EOS as a power series of the variable $w = 1/(1+z) = a/a_0$ around the point $w(z=0) = 1$. The result is

$$\omega(w(z)) = \sum_n \frac{\omega^{(n)}(w=1)}{n!}(w-1)^n = \sum_n \frac{\omega^{(n)}(w=1)}{n!}(-)^n(\frac{z}{1+z})^n. \tag{4}$$

Note that if we expand the EOS as a power series of the variable $y = z/(1+z)$ around $y(z=0) = 0$, we will end up with a similar power series:

$$\omega(y(z)) = \sum_n \frac{\omega^{(n)}(y=0)}{n!} y^n. \tag{5}$$

The Taylor series are normally convergent quickly depending on the structure of the expansion coefficients. Nonetheless, one would prefer to choose a more appropriate expansion parameter in order to make the series converge more rapidly. As a result, leading order terms will be enough to extract the most important physics from the underlying theory. Therefore the advantage of the y expansion is that the power series converges rapidly for all range of the parameter $-1 < y = z/(1+z) < 1$, or equivalently, $-1/2 < z < \infty$ as compared to the range $|z| < 1$ for the power series expansion of z shown in Eq. (3).

One notes, however, that one should also expand all physical quantities and the field equations to the same order of precision we adopted for the EOS expansion. Higher order contributions will not be reliable unless one can show that the higher order terms does not affect the physics very much. For example, the liner order is good enough for the expansion of the EOS modelling SUGRA model [13]. This is because that the linear term fits the predicted EOS for SUGRA model to a very high precision. One readily realizes, however, that it is not easy to track the series expansion order by order due to the form of the Eq. (1) for the EOS for the y expansion. This is because that after performing the integration, one needs to pay attention to the distorted integration result. Indeed, one needs to write $y = 1 - w$ in order to perform the integration involving $d\ln w$. Indeed, one will need to recombine the result back to a power series of y. The trouble is that the lower order terms could sometimes hide in the higher order terms in w. It may not be easy to track clearly the lower order y expansion when we perform the expansion, for example, due to the exponential factor in the Eq. (1). Therefore, one finds that the most natural way to expand the EOS is to expand it in terms of the variable $x = \ln(1+z)$ around the point $x(z=0) = 0$. Indeed, this power series can be shown to be:

$$\omega(x(z)) = \sum_n \frac{\omega^{(n)}(0)}{n!} x^n. \tag{6}$$

And this power series converge very rapidly in the range $-1 < \ln(1+z) < 1$, or equivalently, $-0.63 \sim 1/e - 1 < z < e - 1 \sim 1.72$. Here $e \sim 2.72$ is the natural factor. Note that this limit happens to agree with the proposed scope of the SNAP mission. We will study these different power series expansion for the EOS and its applications in the following sections.

III. SOME PRACTICAL EXPANSIONS

Due to the structure of the differential equations (1), one finds that it will be easy for us to compute related Taylor series by expanding ω either as functions of z, or $y \equiv z/(1+z)$, or $x \equiv \ln z$. Therefore, we will present details of these expansion series in this section. For convenience, we will use repeated notation for the expansion series in this section for convenience and economics of notations. One should bear in mind that these coefficients are defined differently associated with different arguments y, x and z defined in each subsection.

A. Power series of $z/(1+z)$

Note that the expansion parameter $y \equiv z/(1+z) = 1 - a/a_0$. Therefore, the Taylor series expansion around small y is equivalent to a series expansion around $a = a_0$. This is again a series expansion around the very recent universe near $a = a_0$. The expansion series in y has a naive convergence range for all $y < 1$ which is defined to be the early universe with $a < a_0$. We will show in details how to extract the leading terms in the y-expansion with $y = z/(1+z)$ around the point $y(z=0) = 0$. One can write the expansion coefficient as $\omega_n = (1+\omega)^{(n)}(y=0)/n!$ such that the power series for the expansion of the EOS becomes

$$1 + \omega(y(z)) = \sum_n \omega_n y^n. \tag{7}$$

Hence the Eq. (1) can be shown to be

$$d \ln \rho(y) = 3(1+\omega)\frac{dy}{1-y} = d\left[\sum_n \sum_k 3\omega_n \frac{y^{n+k+1}}{n+k+1}\right]. \tag{8}$$

Note that we are now expanding with respect to the smooth function $(1+\omega)$, instead of ω, as a Taylor series for convenience. Therefore, one can integrate above equation to obtain

$$\rho(y) = \rho_0 \exp[\ 3 \sum_n \sum_k \frac{\omega_n y^{n+k+1}}{n+k+1}\ ] \equiv \rho_0 X(y) = \rho_0 \sum_n X_n y^n. \tag{9}$$

Note that one needs to expand the function ρ as a power series of y too in order to extract the approximated solution with appropriate order. The expansion coefficient X_n is defined as $X_n = X^{(n)}(y=0)/n!$. Here, the superscript in X' denotes differentiation with respect to the argument y of the function $X(y)$. One can show that $X' = XY$ with $Y = 3\sum_n \sum_k \omega_n y^{n+k}$. In addition, one can show that

$$Y^{(l)}(y) = 3\ \sum_n \sum_k \frac{(n+k)!}{(n+k-l)!}\omega_n y^{n+k-l}. \tag{10}$$

Hence one has

$$Y^{(l)}(y=0) = 3(l!)\ \sum_{n=0}^{l} \omega_n. \tag{11}$$

One can also show, for example, that

$$X'' = X(Y^2 + Y') \tag{12}$$

$$X''' = X(Y^3 + 3YY' + Y'') \tag{13}$$

$$X^{(4)} = X(Y^4 + 6Y^2Y' + 3(Y')^2 + 4YY'' + Y'''). \tag{14}$$

This series does not appear to have a more compact close form for the multiple differentiation with respect to y. One can, however, put the equations as a more compact format:

$$X^{(l+1)} = X[Y + \frac{d}{dy}]^l Y. \tag{15}$$

It appears, however, that one needs to do it manually even it is straightforward. We will only list the leading terms as this is already suitable expansion for our purpose at this moment.

Hence one has

$$X_0 = 1, \tag{16}$$

$$X_1 = 3\omega_0, \tag{17}$$

$$X_2 = \frac{1}{2}[9\omega_0^2 + 3\omega_0 + 3\omega_1], \tag{18}$$

$$X_3 = \frac{1}{2}[9\omega_0^3 + 9\omega_0(\omega_0+\omega_1) + 2(\omega_0+\omega_1+\omega_2)], \tag{19}$$

$$X_4 = \frac{1}{8}[27\omega_0^4 + 54\omega_0^2(\omega_0+\omega_1) + 9(\omega_0+\omega_1)^2 + 24\omega_0(\omega_0+\omega_1+\omega_2) + 6(\omega_0+\omega_1+\omega_2+\omega_3)]. \tag{20}$$

Therefore, one can expand the final expression for the energy density ρ accordingly. Indeed, the result is

$$\begin{aligned}\rho &= \rho_0\{1+3\omega_0 y+\frac{1}{2}[9\omega_0^2+3\omega_0+3\omega_1]y^2+\frac{1}{2}[9\omega_0^3+9\omega_0(\omega_0+\omega_1)+2(\omega_0+\omega_1+\omega_2)]y^3 \\ &+ \frac{1}{8}[27\omega_0^4+54\omega_0^2(\omega_0+\omega_1)+9(\omega_0+\omega_1)^2+24\omega_0(\omega_0+\omega_1+\omega_2)+6(\omega_0+\omega_1+\omega_2+\omega_3)]y^4\}+O(y^5) \\ &= \rho_0\{\exp[3\omega_0 y]+\frac{3}{2}[\omega_0+\omega_1]y^2+\frac{1}{2}[9\omega_0(\omega_0+\omega_1)+2(\omega_0+\omega_1+\omega_2)]y^3 \\ &+ \frac{1}{8}[54\omega_0^2(\omega_0+\omega_1)+9(\omega_0+\omega_1)^2+24\omega_0(\omega_0+\omega_1+\omega_2)+6(\omega_0+\omega_1+\omega_2+\omega_3)]y^4\}+O(y^5) \end{aligned} \tag{21}$$

to the order of y^4. Note that one keep the order of precision to y^4 in computing the energy density ρ even we are expanding the EOS only to the order of y^3. This is due to the special structure in the energy momentum conservation law (1). In addition, the largest power terms of ω_0 in the series come from the combination $\sum_n [Y^n(0)/n!]y^n = \exp[3\omega_0 y]$ can be summed over directly. Moreover, one can also show that the Hubble parameter $H = H_0 X^{1/2}$ with $H_0 = \sqrt{8\pi G\rho_0/3}$. And the expansion for $X^{1/2}$ can be obtained by replacing all ω_n with $\omega_n/2$ in writing the expansion for X. Therefore one has

$$\begin{aligned}H &= H_0\{\exp[\frac{3}{2}\omega_0 y]+\frac{3}{4}(\omega_0+\omega_1)y^2+\frac{1}{8}[9\omega_0(\omega_0+\omega_1)+4(\omega_0+\omega_1+\omega_2)]y^3+\frac{1}{32}[\\ &\quad 27\omega_0^2(\omega_0+\omega_1)+9(\omega_0+\omega_1)^2+24\omega_0(\omega_0+\omega_1+\omega_2)+12(\omega_0+\omega_1+\omega_2+\omega_3)]y^4\}+O(y^5).\end{aligned} \tag{22}$$

Note also that one can also compute the conformal time according to the expression:

$$H_0\eta = \int_0^z dz' X^{-\frac{1}{2}} = \int_0^y dy' \frac{X^{-\frac{1}{2}}}{(1-y')^2} \tag{23}$$

which comes from the definition $d\eta = dt/a$. Knowing that $1/(1-y)^2 = \sum_n (n+1)y^n$, one can show that

$$H_0\eta = \int_0^y dy' X^{-\frac{1}{2}} \sum_n (n+1)y'^n. \tag{24}$$

One can write $X^{-1/2} = \sum_k x_k y^k = \sum_k X_k(\omega_n \to -\omega_n/2)y^k$, and expand $H_0\eta = \sum_k \eta_k y^k$ for convenience. Therefore, one has

$$\eta_l = \sum_{n=0}^{l-1} \frac{n+1}{l} x_{l-n-1} \tag{25}$$

for $l \geq 1$. Note that $\eta_0 = 0$. As a result, one can easily reconstruct the power series of $H_0\eta$. Therefore, one has

$$\begin{aligned}H_0\eta &= y-[\frac{3}{4}\omega_0-1]y^2+[\frac{3}{8}\omega_0^2-\frac{1}{4}\omega_1-\frac{5}{4}\omega_0+1]y^3-[\frac{9}{64}\omega_0^3-\frac{27}{32}\omega_0^2-\frac{9}{32}\omega_0\omega_1+\frac{13}{8}\omega_0+\frac{1}{2}\omega_1+\frac{1}{8}\omega_2-1]y^4+O(y^5) \\ &= \frac{y}{1-y}-\frac{3}{4}\omega_0 y^2+[\frac{3}{8}\omega_0^2-\frac{1}{4}\omega_1-\frac{5}{4}\omega_0]y^3-[\frac{9}{64}\omega_0^3-\frac{27}{32}\omega_0^2-\frac{9}{32}\omega_0\omega_1+\frac{13}{8}\omega_0+\frac{1}{2}\omega_1+\frac{1}{8}\omega_2]y^4+O(y^5).\end{aligned} \tag{26}$$

Note that terms independent of ω_0, $y+y^2+\cdots$, comes from the combination $K_0 \equiv \sum_{l=1}^\infty \eta_l y^l = \sum_{l=1}^\infty \sum_n^{l-1}(n+1)x_{l-n-1}y^l/l$. Taking the terms with $n = l-1$ in n-summation, one can show that this summation becomes $\sum_{l=1}^\infty x_0 y^l = \sum_{l=1}^\infty y^l = y/(1-y)$.

B. power series of $\ln(1+z)$

Note that the expansion parameter $x \equiv \ln(1+z) = -\ln(a/a_0)$. Therefore, the Taylor series expansion around small $|x|$ is equivalent to a series around $a = a_0$. This is again a series expansion around the very recent universe near $a = a_0$. Indeed, the expansion series in x has a naive convergence range for all $0 < |x| < 1$ which is defined to be the early universe with $a_0/e < a < a_0$. Note that we are interested only in the past universe where $a < a_0$. We will show in details how to extract the leading terms in the x-expansion with $x = \ln(1+z)$ around the point $x(z=0) = 0$. One can write the expansion coefficient as $\omega_n = (1+\omega)^{(n)}(x=0)/n!$ such that the power series for the expansion of the EOS becomes

$$1 + \omega(x(z)) = \sum_n \omega_n x^n. \tag{27}$$

Note that we use the same notation for ω_n in different parameterizations for convenience. Hence the Eq. (1) can be shown to be

$$d \ln \rho(x) = 3(1+\omega)dx = d \, \left[\sum_n 3\omega_n \frac{x^{n+1}}{n+1} \right]. \tag{28}$$

Note that we are now expanding the physical quantities with respect to the function $(1+\omega)$ instead of ω for convenience. Therefore, one can integrate above equation to obtain

$$\rho(x) = \rho_0 \exp[\; 3 \sum_n \frac{\omega_n x^{n+1}}{n+1} \;] \equiv \rho_0 X(x) = \rho_0 \sum_n X_n x^n. \tag{29}$$

Note that one needs to expand the function ρ as a power series of x too in order to extract the approximated solution with appropriate order. The expansion coefficient X_n is defined as $X_n = X^{(n)}(x=0)/n!$. One can show that $X' = XY$ with $Y = 3\sum_n \omega_n x^n = 3(1+\omega)$. Therefore, one has

$$Y^{(l)}(x=0) = 3(l!)\; \omega_l. \tag{30}$$

One can also show, for example, that

$$\begin{aligned} X'' &= X(Y^2 + Y') && (31)\\ X''' &= X(Y^3 + 3YY' + Y'') && (32)\\ X^{(4)} &= X(Y^4 + 6Y^2Y' + 3(Y')^2 + 4YY'' + Y'''). && (33) \end{aligned}$$

In addition, one can show that

$$X^{(l)} = \sum_n \sum_k 3X_n\omega_k \frac{(n+k)!}{(n+k-l+1)!} x^{n+k-l+1}. \tag{34}$$

Therefore, one has

$$X^{(l)}(0) = \sum_n^{l-1} 3X_n \omega_{l-n-1}(l-1)!. \tag{35}$$

Hence one obtains the recurrence relation for the expansion coefficients of X_n:

$$X_l = \frac{3}{l} \sum_{n=o}^{l-1} X_n \omega_{l-n-1}. \tag{36}$$

As a result, one has, for example,

$$\begin{aligned} X_0 &= 1, && (37)\\ X_1 &= 3\omega_0, && (38)\\ X_2 &= \frac{1}{2}[9\omega_0^2 + 3\omega_1], && (39)\\ X_3 &= \frac{1}{2}[9\omega_0^3 + 9\omega_0\omega_1 + 2\omega_2], && (40)\\ X_4 &= \frac{1}{8}[27\omega_0^4 + 54\omega_0^2\omega_1 + 9\omega_1^2 + 24\omega_0\omega_2 + 6\omega_3]. && (41) \end{aligned}$$

Therefore, one can expand the final expression for the energy density ρ accordingly. Indeed, one has

$$\begin{aligned}\rho &= \rho_0\{1+3\omega_0 x+\frac{1}{2}[9\omega_0^2+3\omega_1]x^2+\frac{1}{2}[9\omega_0^3+9\omega_0\omega_1+2\omega_2]x^3 \\ &+ \frac{1}{8}[27\omega_0^4+54\omega_0^2\omega_1+9\omega_1^2+24\omega_0\omega_2+6\omega_3]x^4\}+O(x^5) \\ &= \rho_0\left\{\exp[3\omega_0 x]+\frac{3}{2}\omega_1 x^2+\frac{1}{2}[9\omega_0\omega_1+2\omega_2]x^3+\frac{1}{8}[54\omega_0^2\omega_1+9\omega_1^2+24\omega_0\omega_2+6\omega_3]x^4\right\}+O(x^5).\end{aligned} \tag{42}$$

In addition, one can show that the Hubble parameter $H = H_0X^{1/2}$ with $H_0 = \sqrt{8\pi G\rho_0/3}$. And the expansion for $X^{1/2}$ can be obtained by replacing all ω_n with $\omega_n/2$ in writing the expansion for X. The result is

$$H = H_0\left\{\exp[\frac{3}{2}\omega_0 x]+\frac{3}{4}\omega_1 x^2+\frac{1}{8}[9\omega_0\omega_1+4\omega_2]x^3+\frac{1}{32}[27\omega_0^2\omega_1+9\omega_1^2+24\omega_0\omega_2+12\omega_3]x^4\right\}+O(x^5). \tag{43}$$

Note also that one can also compute the conformal time according to the expression:

$$H_0\eta = \int_0^z dz' X^{-\frac{1}{2}} = \int_0^x dx' X^{-\frac{1}{2}}\exp[x'] = \sum_n \int_0^x dx' X^{-\frac{1}{2}}\frac{x'^n}{n!}. \tag{44}$$

Therefore, one can easily compute the expansion of η in a straightforward manner. One can write $X^{-1/2} = \sum_k x_k x^k = \sum_k X_k(\omega_n \to -\omega_n/2)x^k$, and $H_0\eta = \sum_k \eta_k x^k$ for convenience. Therefore, one has

$$\eta_l = \sum_{n=0}^{l-1}\frac{x_{l-n-1}}{l\, n!} \tag{45}$$

for all $l \geq 1$. Note that $\eta_0 = 0$. As a result, one can easily reconstruct the power series of $H_0\eta$. Indeed, one has

$$\begin{aligned}H_0\eta &= x+\frac{1}{4}(2-3\omega_0)x^2+\frac{1}{24}(9\omega_0^2-6\omega_1-12\omega_0+4)x^3 \\ &- \frac{1}{192}(27\omega_0^3-54\omega_0^2-54\omega_0\omega_1+36\omega_0+36\omega_1+24\omega_2-8)x^4+O(x^5) \\ &= e^x-1-\frac{3}{4}\omega_0x^2+\frac{1}{8}(3\omega_0^2-2\omega_1-4\omega_0)x^3-\frac{1}{64}(9\omega_0^3-18\omega_0^2-18\omega_0\omega_1+12\omega_0+12\omega_1+8\omega_2)x^4+O(x^5).\end{aligned} \tag{46}$$

Note that a trivial complete sum of terms $x+x^2/2!+x^3/3!\cdots = e^x-1$, derived from the $n = l-1$ terms in Eq. (45), has been summed over for convenience in the final expression.

C. power series of z

Note that the expansion parameter $z \equiv (a_0/a)-1$. Hence the range of z of interest is the range $z < 1$ in the past universe which is equivalent to the range where $a_0/2 < a < a_0$. Therefore, the Taylor series expansion around small $|z|$ is equivalent to a series around $a = a_0$. This is again a series expansion around the very recent universe near $a = a_0$. We will show in details how to extract the leading terms in the z-expansion around the point $z = 0$. One can write the expansion coefficient as $\omega_n = (1+\omega)^{(n)}(z=0)/n!$ such that the power series for the expansion of the EOS becomes

$$1+\omega(z) = \sum_n \omega_n z^n. \tag{47}$$

Note that we use the same notation for ω_n in different parameterizations for convenience. Hence the Eq. (1) can be shown to be

$$d\ln\rho(z) = 3(1+\omega)\frac{dz}{1+z} = d\left[\sum_{n,k}(-1)^k 3\omega_n\frac{z^{n+k+1}}{n+k+1}\right]. \tag{48}$$

Note that we are now expanding the physical quantities with respect to the function $(1+\omega)$ instead of ω for convenience. Therefore, one can integrate above equation to obtain

$$\rho(z) = \rho_0 \exp\left[3\sum_{n,k}(-1)^k \frac{\omega_n z^{n+k+1}}{n+k+1} \right] \equiv \rho_0 X(z) = \rho_0 \sum_n X_n z^n. \tag{49}$$

Note that one needs to expand the function ρ as a power series of z too in order to extract the approximated solution with appropriate order. The expansion coefficient X_n is defined as $X_n = X^{(n)}(z=0)/n!$. One can show that $X' = XY$ with $Y = 3\sum_{n,k}(-1)^k \omega_n z^{n+k}$. Therefore, one has

$$Y^{(l)}(z) = 3\ \sum_n \sum_k (-1)^k \frac{(n+k)!}{(n+k-l)!} \omega_n z^{n+k-l}. \tag{50}$$

Hence one has

$$Y^{(l)}(z=0) = 3\ \ l!\ \sum_{n=0}^{l} (-1)^{n+l} \omega_n. \tag{51}$$

One can also show, for example, that

$$X'' = X(Y^2 + Y') \tag{52}$$
$$X''' = X(Y^3 + 3YY' + Y'') \tag{53}$$
$$X^{(4)} = X(Y^4 + 6Y^2Y' + 3(Y')^2 + 4YY'' + Y'''). \tag{54}$$

This series does not appear to have a more compact close form for the multiple differentiation with respect to z. One can, however, put the equations as a more compact format:

$$X^{(l+1)} = X[Y + \frac{d}{dz}]^l Y. \tag{55}$$

It appears, however, that one needs to do it manually even it is straightforward. We will only list the leading terms as this is already suitable expansion for our purpose at this moment.

Hence one has

$$X_0 = 1, \tag{56}$$
$$X_1 = 3\omega_0, \tag{57}$$
$$X_2 = \frac{1}{2}[9\omega_0^2 - 3\omega_0 + 3\omega_1], \tag{58}$$
$$X_3 = \frac{1}{2}[9\omega_0^3 + 9\omega_0(\omega_1 - \omega_0) + 2(\omega_0 - \omega_1 + \omega_2)], \tag{59}$$
$$X_4 = \frac{1}{8}[27\omega_0^4 + 54\omega_0^2(\omega_1 - \omega_0) + 9(\omega_0 - \omega_1)^2 + 24\omega_0(\omega_0 - \omega_1 + \omega_2) + 6(\omega_3 - \omega_2 + \omega_1 - \omega_0)]. \tag{60}$$

Therefore, one can expand the final expression for the energy density ρ accordingly. Indeed, the result is

$$\begin{aligned} \rho &= \rho_0\{\exp[3\omega_0 z] - \frac{3}{2}[\omega_0 - \omega_1]z^2 + \frac{1}{2}[9\omega_0(\omega_1 - \omega_0) + 2(\omega_0 - \omega_1 + \omega_2)]z^3 \\ &+ \frac{1}{8}[54\omega_0^2(\omega_1 - \omega_0) + 9(\omega_0 - \omega_1)^2 + 24\omega_0(\omega_0 - \omega_1 + \omega_2) + 6(\omega_3 - \omega_2 + \omega_1 - \omega_0)]z^4\} + O(z^5) \end{aligned} \tag{61}$$

to the order of z^4. Note that one keep the order of precision to z^4 in computing the energy density ρ even we are expanding the EOS only to the order of z^3. This is due to the special structure in the energy momentum conservation law (1). In addition, one can show that the Hubble parameter $H = H_0 X^{1/2}$ with $H_0 = \sqrt{8\pi G\rho_0/3}$. And the expansion for $X^{1/2}$ can be obtained by replacing all ω_n with $\omega_n/2$ in writing the expansion for X. Therefore one has

$$\begin{aligned} H &= H_0\{\exp[\frac{3}{2}\omega_0 z] + \frac{3}{4}(\omega_1 - \omega_0)z^2 + \frac{1}{8}[9\omega_0(\omega_1 - \omega_0) + 4(\omega_0 - \omega_1 + \omega_2)]z^3 \\ &+ \frac{1}{32}[27\omega_0^2(\omega_1 - \omega_0) + 9(\omega_0 - \omega_1)^2 + 24\omega_0(\omega_0 - \omega_1 + \omega_2) + 12(\omega_3 - \omega_2 + \omega_1 - \omega_0)]z^4\} + O(z^5). \end{aligned} \tag{62}$$

Note also that one can also compute the conformal time according to the expression:

$$H_0\eta = \int_0^z dz' X^{-\frac{1}{2}}. \tag{63}$$

One can write $X^{-1/2} = \sum_k x_k z^k = \sum_k X_k(\omega_n \to -\omega_n/2) z^k$ for convenience. Therefore, one has

$$H_0\eta = \sum_{n=1}^{\infty} \frac{x_{k-1}}{k} z^k \tag{64}$$

As a result, one can easily reconstruct the power series of $H_0\eta$. Therefore, one has

$$\begin{aligned} H_0\eta &= z - \frac{3\omega_0}{4}z^2 + \frac{1}{8}[3\omega_0^2 + 2\omega_0 - 2\omega_1]z^3 - \frac{1}{64}[9\omega_0^3 + 18\omega_0^2 - 18\omega_0\omega_1 + 8(\omega_0 - \omega_1 + \omega_2)]z^4 + O(z^5) \\ &= \frac{2}{3\omega_0}(1 - \exp[-\frac{3\omega_0}{2}z]\ ) + \frac{1}{4}[\omega_0 - \omega_1]z^3 - \frac{1}{16}[3\omega_0^2 - 3\omega_0\omega_1 + 4(\omega_0 - \omega_1 + \omega_2)]z^4 + O(z^5). \end{aligned} \tag{65}$$

IV. COMPARISON AND ANALYSIS

In summary, one has obtained the series sum of the corresponding energy density, Hubble parameter and conformal time for different parameterizations of the function ω. For $y = z/(1+z)$ one has:

$$\begin{aligned} \rho(y) &= \rho_0\{\exp[3\omega_0 y] + \frac{3}{2}[\omega_0 + \omega_1]y^2 + \frac{1}{2}[9\omega_0(\omega_0 + \omega_1) + 2(\omega_0 + \omega_1 + \omega_2)]y^3 \\ &+ \frac{1}{8}[54\omega_0^2(\omega_0 + \omega_1) + 9(\omega_0 + \omega_1)^2 + 24\omega_0(\omega_0 + \omega_1 + \omega_2) + 6(\omega_0 + \omega_1 + \omega_2 + \omega_3)]y^4\} + O(y^5) \end{aligned} \tag{66}$$

$$\begin{aligned} H(y) &= H_0\{\exp[\frac{3}{2}\omega_0 y] + \frac{3}{4}(\omega_0 + \omega_1)y^2 + \frac{1}{8}[9\omega_0(\omega_0 + \omega_1) + 4(\omega_0 + \omega_1 + \omega_2)]y^3 + \frac{1}{32}[\\ &27\omega_0^2(\omega_0 + \omega_1) + 9(\omega_0 + \omega_1)^2 + 24\omega_0(\omega_0 + \omega_1 + \omega_2) + 12(\omega_0 + \omega_1 + \omega_2 + \omega_3)]y^4\} + O(y^5). \end{aligned} \tag{67}$$

$$H_0\eta(y) = \frac{y}{1-y} - \frac{3}{4}\omega_0 y^2 + [\frac{3}{8}\omega_0^2 - \frac{1}{4}\omega_1 - \frac{5}{4}\omega_0]y^3 - [\frac{9}{64}\omega_0^3 - \frac{27}{32}\omega_0^2 - \frac{9}{32}\omega_0\omega_1 + \frac{13}{8}\omega_0 + \frac{1}{2}\omega_1 + \frac{1}{8}\omega_2]y^4 + O(y^5) \tag{68}$$

For $x = \ln(1+z)$, one has:

$$\rho(x) = \rho_0\left\{\exp[3\omega_0 x] + \frac{3}{2}\omega_1 x^2 + \frac{1}{2}[9\omega_0\omega_1 + 2\omega_2]x^3 + \frac{1}{8}[54\omega_0^2\omega_1 + 9\omega_1^2 + 24\omega_0\omega_2 + 6\omega_3]x^4\right\} + O(x^5). \tag{69}$$

$$H(x) = H_0\left\{\exp[\frac{3}{2}\omega_0 x] + \frac{3}{4}\omega_1 x^2 + \frac{1}{8}[9\omega_0\omega_1 + 4\omega_2]x^3 + \frac{1}{32}[27\omega_0^2\omega_1 + 9\omega_1^2 + 24\omega_0\omega_2 + 12\omega_3]x^4\right\} + O(x^5). \tag{70}$$

$$H_0\eta(x) = \mathrm{e}^x - 1 - \frac{3}{4}\omega_0 x^2 + \frac{1}{8}(3\omega_0^2 - 2\omega_1 - 4\omega_0)x^3 - \frac{1}{64}(9\omega_0^3 - 18\omega_0^2 - 18\omega_0\omega_1 + 12\omega_0 + 12\omega_1 + 8\omega_2)x^4 + O(x^5) \tag{71}$$

For the z expansion, one has:

$$\begin{aligned} \rho(z) &= \rho_0\{\exp[3\omega_0 z] - \frac{3}{2}[\omega_0 - \omega_1]z^2 + \frac{1}{2}[9\omega_0(\omega_1 - \omega_0) + 2(\omega_0 - \omega_1 + \omega_2)]z^3 \\ &+ \frac{1}{8}[54\omega_0^2(\omega_1 - \omega_0) + 9(\omega_0 - \omega_1)^2 + 24\omega_0(\omega_0 - \omega_1 + \omega_2) + 6(\omega_3 - \omega_2 + \omega_1 - \omega_0)]z^4\} + O(z^5) \end{aligned} \tag{72}$$

$$\begin{aligned} H(z) &= H_0\{\exp[\frac{3}{2}\omega_0 z] + \frac{3}{4}(\omega_1 - \omega_0)z^2 + \frac{1}{8}[9\omega_0(\omega_1 - \omega_0) + 4(\omega_0 - \omega_1 + \omega_2)]z^3 \\ &+\frac{1}{32}[27\omega_0^2(\omega_1 - \omega_0) + 9(\omega_0 - \omega_1)^2 + 24\omega_0(\omega_0 - \omega_1 + \omega_2) + 12(\omega_3 - \omega_2 + \omega_1 - \omega_0)]z^4\} + O(z^5). \end{aligned} \tag{73}$$

$$H_0\eta(z) = \frac{2}{3\omega_0}(1 - \exp[-\frac{3\omega_0}{2}z]\ ) + \frac{1}{4}[\omega_0 - \omega_1]z^3 - \frac{1}{16}[3\omega_0^2 - 3\omega_0\omega_1 + 4(\omega_0 - \omega_1 + \omega_2)]z^4 + O(z^5). \tag{74}$$

Note that the expansion coefficients ω_n are defined differently for different expansions even we are using the same notation for convenience. Also note that at small red-shift $z \ll 1$, y and x are both very close to z. This is the reason why we are ending up with similar leading term in each expansion. Once the red-shift z is extended, difference in the leading terms will be significant. One is therefore able to distinguish the contribution of the next-leading terms.

As mentioned earlier, one should be able to determine the expanding coefficients ω_n with the result from the measurements of future experiments. Once the expansion coefficients are determined from fitting with the Hubble parameter or energy density, one will be able to reconstruct the function ω to higher order and higher precision. The result should be helpful to determine the origin and the nature of the matter sources and to be compared with some possible combinations of various fundamental theories.

The linear parametrization of the ω has been applied to the study of the evolution history of our universe. The linear models adopted has been shown to be useful in making predictions for future observations [13]. One is naturally lead to answer the question whether one should take the linear model as a complete theory and integrate it without worrying about the higher order corrections. To be more specific, linear model like $\omega(z) = \omega_0' + \omega_1 z$ has been taken as a complete theory. Related physical quantities are computed with all power of ω_1, in the power series shown above, summed over for the final result. On the other hand, if the linear model is taken as leading order approximation, one should ignore higher order contributions from $z^n (\forall n \geq 2)$.

If higher order corrections is small and negligible, it does not matter much whether one ignores or includes all higher order (in ω_1) corrections. For example, in a convergent series, higher order contributions are smaller and smaller order by order. Therefore, it is fine to sum over all terms related to the expansion parameter ω_1. Otherwise, one should pay attention to the convergent properties of these expansions for possible deviations.

Looking at our results, it is easy to find that a re-summation of the expansion coefficients ω_n in these formulae is difficult to obtain except the re-summation of ω_0 which has been already shown in above equations. Fortunately, one can do it by a different way which will be shown in a moment. As a result, one would be able to see whether an expansion in power of ω_n is reliable or not.

In next section, we will show how to sum up all terms order by order as series of expansion coefficients ω_n. Taking the linear model $\omega = \omega_0' + \omega_1 y$ as a complete theory and include all effect of ω_0' and ω_1 include more effect from the higher order y^n contribution as compared to the leading order approximation approach. The inclusion of these effect may not be reliable unless they are small as compared to the leading terms.

In addition, taking linear model as a complete theory, the effects from all higher order in $\omega_{n>1}$ are ignored. The truncation may need to appreciable error in evaluation of related physical observables unless these higher order terms are also small. We will also analyze the error due to the contributions from the higher order $\omega_{n>1}$ term.

Note again that linear model may serve as a good approximation theory in y expansion because the naive convergence range $y < 1$ covers the entire history of our universe from $z = 0 \to \infty$. If the linear model fits the original theory very well at small y and higher order contributions are not appreciable, linear model can be extended to remain valid even at larger y. The higher order effect related to ω_2 will be evaluated also for the hope that future experiments may provide better resolution to distinguish possible minor effect.

In addition, one more advantage of the linear model written as $\omega = \omega_0' + \omega_1 y$ over the linear model in expansion of z can be easily seen from the relation:

$$\omega(y) = \omega_0' + \omega_1 y = \omega_0' + \omega_1 \frac{z}{1+z} = \omega_0' + \omega_1 z(1 - z + z^2 - \cdots) = \omega(z) - \omega_1 z^2 (1 - z + z^2 + \cdots). \tag{75}$$

Indeed, one can see that linear model $\omega(y)$ contains a few more higher order $z^{n>1}$ effect than the linear model $\omega(z) \equiv \omega_0' + \omega_1 z$. The effect of the higher order terms depends, however, on the actual deviation from the underlying theory like SUGRA or inverse power law models. Evidences show that linear model in y works better than the linear model in z at large redshift for both reasons.

V. ERROR ESTIMATION OF THE LINEAR MODEL

Note that the expansion parameter $y \equiv z/(1+z) = 1 - a/a_0$. Therefore, the Taylor series expansion around small y is equivalent to a series expansion around $a = a_0$. This is again a series expansion around the very recent universe near $a = a_0$. The expansion series in y has a naive convergence range for all $y < 1$ which is defined to be the complete early universe with $a < a_0$. We will show in details how to extract the leading terms in the y-expansion with $y = z/(1+z)$ around the point $y(z = 0) = 0$.

Indeed, one can write the expansion coefficient as $\omega_n = (1+\omega)^{(n)}(y = 0)/n!$ such that the power series for the expansion of the EOS becomes $1 + \omega(y(z)) = \sum_n \omega_n y^n$. One can write $w = 1 - y$ and write it as

$$1 + \omega(y) = \sum_n \omega_n y^n = \omega_0 + \sum_{n=1}^{\infty} \omega_n (1-w)^n = \omega(0) + \sum_{n=1}^{\infty} \sum_{k=1}^{n} C_k^n \omega_n (-w)^k. \tag{76}$$

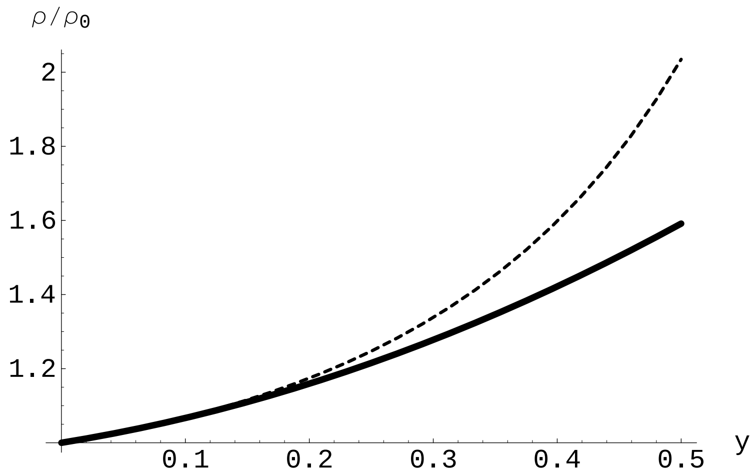

FIG. 1: Thin dotted line plots $\rho_1(y)/\rho_0$ and thick solid lines represents $\rho^1(y)/\rho_0$ for the linear model $\omega = -0.82 + 0.58y$.

Here $\omega(0) = \sum_{n=0}^{\infty} \omega_n = [1 + \omega(y)]_{y=0}$. Hence the Eq. (1) can be shown to be

$$d \ln \rho(y) = 3(1+\omega)\frac{dy}{1-y} = -3(1+\omega)\frac{dw}{w} = -3d\left[\omega(0)\ln w + \sum_{n=1}^{\infty}\sum_{k=1}^{n} C_k^n \frac{\omega_n(-w)^k}{k}\right]. \tag{77}$$

Therefore, one can show that

$$\rho(y) = \rho(0)\left[(1-y)^{-3\omega(o)} \exp[-3\sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{l=1}^{k} C_k^n C_l^k \frac{\omega_n y^l}{k}]\right]. \tag{78}$$

In fact, one can show that when $y \to 1$, the energy density shown in Eq. (78) is dominated and proportional to

$$\rho(y \to 1) \propto (1-y)^{-3\omega(o)} \tag{79}$$

implying $\rho \propto (1-y)^{-3(0.18+\omega_2+\omega_3+\cdots)}$ for the model approximated by $\omega = -0.82 + 0.58y$. This term will diverge at $y = 1$ if all higher order coefficients are small as compared to the leading coefficients such that the sum $\omega(0)$ remain positive. Therefore, one can expand the density as a power series of ω_n from above equation. For comparison, one can show that

$$\rho_2(y) = \rho_1(y)\left[(1-y)\exp[y + \frac{y^2}{2}]\right]^{-3\omega_2} \tag{80}$$

keeping terms to the order of ω_2. Here $\rho_1(y)$ denotes $\rho(y)$ with $\omega_n = 0$ for all $n \geq 2$. Similarly, $\rho_2(y)$ denotes $\rho(y)$ with $\omega_n = 0$ for all $n \geq 3$. Hence the last factor of above equation provides the ω_2 corrections of the power series.

Note that linear model $\omega = -0.82 + 0.58y$ is considered as a complete model with the corresponding energy density $\rho = \rho_1$ shown in Eq. (80). As a complete model, all powers of y^n are included in the evaluation of energy density ρ_1. Indeed, one can show that

$$\rho_1 = \rho_0(1-y)^{-3(\omega_0+\omega_1)}\exp[-3\omega_1 y], \tag{81}$$

$$\rho^1 = \rho_0\{1 + 3\omega_0 y + \frac{1}{2}[9\omega_0^2 + 3\omega_0 + 3\omega_1]y^2\} \tag{82}$$

with ρ^1 given above denoting the truncated energy density to the order of y^2 for the approximated counterpart. Results shown in Fig. 1 indicates that the difference between ρ_1 and ρ^1 is small when y is small. But it becomes appreciable at larger y. Note again that $\omega_0 \equiv 1 + \omega(y=0) = 0.18$ in our notation for the linear model $\omega = -0.82 + 0.58y$.

Similarly, one can show that $H^2 = 8\pi G\rho/3$ can be integrated to give:

$$H(y) = H(0)\left[(1-y)^{-3\omega(o)/2}\exp[-\frac{3}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{l=1}^{k} C_k^n C_l^k \frac{\omega_n y^l}{k}]\right]. \tag{83}$$

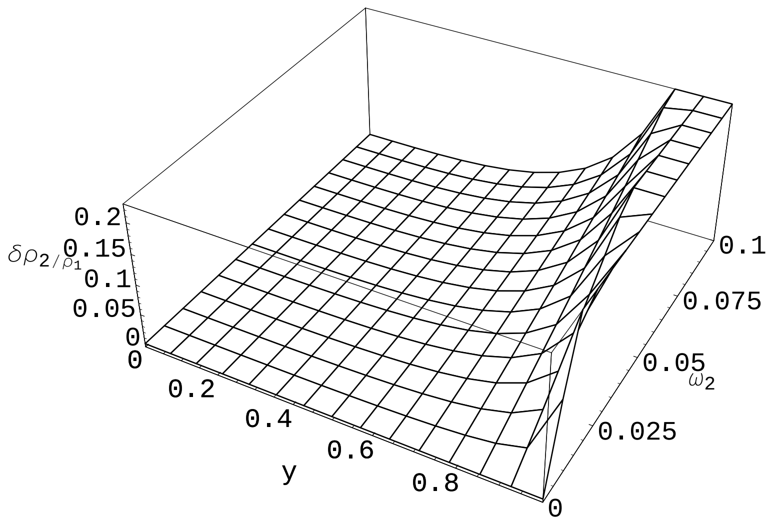

FIG. 2: The error $\rho_2/\rho_1 - 1$ due to the presence of the ω_2 correction, as compared to the linear model, is plotted in 3D format as a function of y and ω_2.

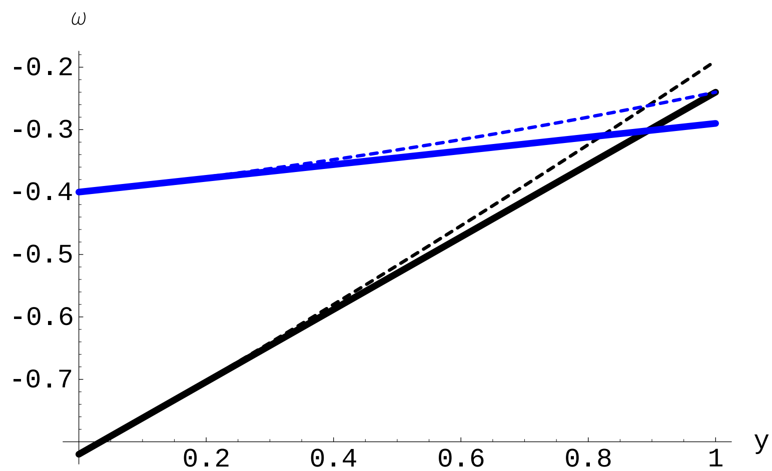

FIG. 3: The equation of state ω is plotted as a function of y. Lower thick solid line denotes the contribution of linear model $\omega = -0.82 + 0.58y$. Lower thin dotted line represents the addition of the y^2 correction with $\omega_2 y^2 = 0.1y^2$. Similarly, Upper thick solid line denotes the contribution of linear model $\omega = -0.4 + 0.11y$. Upper thin dotted line represents the corresponding y^2 correction from $\omega_2 y^2 = 0.1y^2$.

Therefore, one can also expand the density as a power series of ω_n from above equation. Indeed, one can show that

$$H_2(y) = H_1(y)\left[(1-y)\exp[y + \frac{y^2}{2}]\right]^{-3\omega_2/2}. \tag{84}$$

Here $H_1(y)$ denotes $H(y)$ with $\omega_n = 0$ for all $n \geq 2$. Similarly, $H_2(y)$ denotes $H(y)$ with $\omega_n = 0$ for all $n \geq 3$. Hence the last factor of above equation provides the ω_2 corrections of the power series.

In Fig. 2, the error $\rho_2/\rho_1 - 1$ due to the presence of the ω_2 correction, as compared to the linear model, is plotted in 3D format as a function of y and ω_2. It is easy to see that the error becomes appreciable when $y > 7$ and $\omega_2 > 0.4$.

In Fig. 3, the equation of state ω is plotted as a function of y. Lower thick solid line denotes the contribution of linear model $\omega = -0.82 + 0.58y$[19]. This linear model fits SUGRA equation of state very well at small y. Lower thin dotted line represents the addition of the y^2 correction with $\omega_2 y^2 = 0.1y^2$. In addition, the upper thick solid line denotes the contribution of linear model $\omega = -0.4 + 0.11y$[19]. This model fits the Inverse power law model very well at small y. Upper thin dotted line represents the corresponding y^2 correction from $\omega_2 y^2 = 0.1y^2$. Note that it has been shown from Figure 1 of Reference [19], linear models shown above fits the SUGRA and inverse power law model very well at small y as compared to the linear model in z. ω as a

It is easily seen that a small y^2 correction with $\omega_2 = 0.1$ does not modify the function ω very much. In addition, the correction to the density function for $\omega_2 = 0.1$ is not appreciable according to Fig. 2. Therefore, the linear model can be truncated at the order of ω_1 for small y without affecting the final result in an appreciable way.

From above presentations, one also finds that there are two comparisons required for the reliability for the linear model. First of all, one should check if the contributions from higher order coefficients $\omega_{n>1}$ will affect the evaluation results. Secondly, one should also check if the inclusion of higher order terms $y^{n>2}$ in the physical functions, e.g. ρ_1, will affect the final result in an appreciable way.

Due to the fact that the naive convergence range of $y = z/(1+z)$ expansion covers the entire history of our early universe, this expansion appears to be the best way to approximate our physical models. As a result, the approximated linear model may represent the complete theory effectively for a larger domain y provided that the higher order corrections derived from ω_2 is small. Our results indicates that this appears to be the case for the SUGRA models and inverse power law models[19]. Therefore, the corresponding linear model appears to be more reliable than the other linear models expanded in z or $\ln(1+z)$.

Detailed analysis indicates that the linear model is a very reliable approximation within small y. Beyond the region of small y, one should be very careful with the higher order contributions when comparing with the future experiments. In practice, the sources of equation of state ω may be very complicate. In spite of the fact that it may be derived from complicate combinations of many different sources, linear model provides an easy way to determine the leading coefficients that should be very reliable at small y corresponding to our very recent universe. If one is able to determine the next leading term coefficients ω_2 from better precision measurements in the near future, it would provide better way to distinguish the nature of the source of equation of state.

VI. CONCLUSION

Linear models, e.g. $\omega = \omega_0' + \omega_1 y$, should be thought of as leading approximation of a complete Taylor series. If this is so, result derived from higher order in y, e.g. $O(y^2)$ terms, should only provide marginal contribution and hence can be ignored. If the higher order contribution is appreciable, one should be very careful in dealing with the truncated series. As a practical analysis, one should compute all related physical functions also as a power series order by order to obtain reliable observable according to the precision requirement. Otherwise, unphysical contributions may build up and leave the final result invalid. Fortunately, this complication may be unnecessary if the higher order contributions are small as compared to the leading order contribution.

Indeed, linear models of equation of state are considered as effective models for some underlying theoretical models. Many theories including Sugra and Inverse power law model can approximated by linear model described by $\omega = \omega_0' + \omega_1 y$. Therefore these linear models are potentially good candidate to simulate the large redshift properties of the underlying theories because the higher order corrections could be negligible at larger z, corresponding to $y \equiv z/(1+z) \to 1$.

Since $z \to \infty$ is equivalent to $y \to 1$. Hence y-expansion with $y \to 1$ is still in the range of naive convergence of the corresponding Taylor series. Therefore, as long as the higher order terms are negligible as compared to the liner approximation, one is free to consider the linear model as a well-behaved representation of the underlying theory. Otherwise, one should pay attention to possible deviations derived from the higher order corrections.

We have calculated two possible errors when one treats the linear model as a complete theory instead of the leading approximation for an underlying theory. One possible error comes from the inclusion of all higher order terms (in y^n) related to ω_0 and ω_1 in the linear model calculation. The other error comes from the truncated higher order terms in ω_n for the underlying theory. They are both computed and compared in section V.

We have also tried to analyze possible deviations for the Sugra and inverse power law models in this section. Results show that linear model $\omega = \omega_0' + \omega_1 y$ for these two models appears to be a good approximation even at large redshift region as long as the higher order corrections are small.

Since we are only able to determine only a few leading coefficients in the future experiments, linear model appears to serve this purpose very well. Hopefully, one should, however, be able to determine for example the deviation due to the ω_2 contributions and further the understanding of the underlying theory if better resolution can be made possible in the future experiments .

In addition, we have also tried to provide a more complete list of the Taylor series in this paper. Hope that these presentations can provide better information for the quest of the mapping of our early universe.

Therefore, we try to extend the parametrization further to include the effect of higher orders of the series expansion results. Related physical quantities are treated carefully order by order in order to extract more reliable information from these expansions. One also tries to determine the expansion coefficients from the fitting of the measured Hubble parameters or energy density. As a result, one may reconstruct the series expansion of the equation of state and probe the nature and origin of the matter sources.

As mentioned above, we also present an error analysis to find the range of convergence and possible error control for a meaningful truncation. The linear model given by $\omega = -0.82 + 0.58y$ is known to fit the SUGRA result to a good precision at large z even the linear model is already off 27% when $z \sim 1.7$. One shows that the next leading expansion coefficient with $\omega_2 y^2 < 0.1y^2$ only provides at most 20% error to the density function even when the redshift is close to 1100. The error is even down to 10% for the Hubble function H. As a result, one show that a reliable truncation is possible when the higher order expansion coefficient is small for all range $y = z/(z+1) = 0 \to 0.9991$ up to the last scattering surface when $z \sim 1100$. Therefore, it does not matter much whether one sums up all power of y coupled to the leading coefficients ω_0 and ω_1 as long as the higher order corrections due to ω_2 and $\omega_{n>2}$ are negligible.

In spite of the fact that equation of state ω may be derived from complicate combinations of many different sources, linear model provides an easy way to determine the leading coefficients that should be very reliable at small y corresponding to our very recent universe. If one is able to determine the next leading term coefficients ω_2 from better precision measurements in the near future, it would provide better way to distinguish the nature of the source of equation of state.

Indeed, the equation of state can be described by $\omega = p/\rho = (\dot{\phi}^2/2 - V)/(\dot{\phi}^2/2 + V)$ for many different effective theory with effective potential V and scalar field ϕ. For example, $V_{INV} \sim \phi^{-\alpha}$ and $V_{SUGRA} \sim \phi^{-\alpha} \exp[\phi^2]$ for inverse power law models and SUGRA models respectively[19]. The nature of our physical universe may also consist of many other combinations of scalar fields contributions. It is therefore important for us to figure out which model plays the most important role in the expansion history of our universe. The linear model or leading approximation appears to be of help in determining the essential part of the nature of the equation of state and its corresponding effective theory. Therefore a more detailed analysis on the convergence properties of various approximated models is very important as a research topic.

In summary, one has evaluated the series expansion for the energy density, Hubble constant and the conformal time for the y, x and z expansion of the corresponding equation of state function ω up to order four in previous section. The proposed Supernova/Acceleration Probe (SNAP) will carry out observations aiming to determine the equations of state of the energy density, providing insights into the cosmological model, the nature of the accelerating dark energy, and potential clues to fundamental high energy physics theories and gravitation. As a result, we show all suitable ways to parameterizing the equation of state for application to study its effect on the expansion history of the recent universe.

A detailed discussion on the choices of the expansion parameters for the Taylor series of the equation of states ω is presented in this paper accordingly. For example, the Taylor series of the EOS is expanded as power series of the variables $y = z/(1+z)$, $x = \ln(1+z)$ and z respectively. Due to the fact that the naive convergence range of $y = z/(1+z)$ expansion covers the entire history of our early universe, this expansion appears to be the best way to approximate our physical models. As a result, the approximated linear model may represent the complete theory effectively for a larger domain y provided that the higher order corrections derived from ω_2 is small. This appears to be the case for the SUGRA models and inverse power law models. Therefore, the corresponding linear model appears to be more reliable than the other linear models expanded in z or $\ln(1+z)$.

We also show how to obtain the power series for the energy density function, the Hubble parameter and related physical quantities of interest. The method presented here will have significant application in the precision distance-redshift observations aimed to map out the recent expansion history of the universe, including the present acceleration and the transition to matter dominated deceleration.

Since we can power-expand all smooth EOS into a convergent power series for a reasonable range of the expansion parameters, it is more practical for the future probe to determine the expansion coefficients ω_n or equivalently the local derivatives of the EOS. One may need to use different expansion series depending on the convergent speed of the power series. For example, it appears that the leading order term in the y expansion is better as a nice result for the SUGRA prediction. This is because the leading term is close enough to the theoretical prediction at small y region[13]. Nonetheless, one expects that fitting for a few more leading terms in the Taylor series will be able to provide us better information about the nature of the function ω. In addition, the result shown here is independent of the choice of the time t_0. The local measurement of the expansion coefficients can be extended to the comparison of the expansion coefficients at any time.

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